# Nash Networks with Heterogeneous Agents[1]

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**Abstract**

A non-cooperative model of network formation is developed. Agents form links with others based on the cost of the link and its assessed benefit. Link formation is one-sided, i.e., agents can initiate links with other agents without their consent, provided the agent forming the link makes the appropriate investment. Information flow is two-way. The model builds on the work of Bala and Goyal, but allows for agent heterogeneity. Whereas they permit links to fail with a certain common probability, in our model the probability of failure can be different for different links. We investigate Nash networks that exhibit connectedness and super-connectedness. An explicit characterization of certain star networks is provided. Efficiency, Pareto-optimality, and existence issues are discussed through examples. We explore four alternative model specifications to address potential shortcomings.

*JEL Classification: D82, D83*

# 1 Introduction

The *Internet* provides ample testimony to the fact that information dissemination affects all aspects of economic activity. It is creating globalization that has hitherto been unprecedented in human history. Nowadays fashions and fads emerging in one country are easily communicated across the world with almost no time lag. Financial troubles in one country now have devastating consequences for other economies as the contagion moves across boundaries with relative ease. Yet, the East Asian financial crisis also demonstrated that economies where information networks were relatively primitive remained largely insulated from the crisis. This indicates that both the structure and the technology of information dispersion are important determinants of its consequences.

In the present paper, we look at the formation of social networks which serve as a mechanism for information transmission. The structural aspects of information dissemination are modelled by means of a social network. The role of technology is studied by examining the reliability of the network. Social networks have played a vital role in the diffusion of information across society in settings as diverse as referral networks for jobs (Granovetter (1974) and Loury (1977)) and in assessing quality of products ranging from cars to computers (Rogers and Kincaid (1981)). Information in most societies can either be obtained in the market-place or through a non-market environment like a social network. For instance, in developed countries credit agencies provide credit ratings for borrowers, while in many developing countries credit worthiness is assessed through a social network organized along ethnic lines.

Agents in our model are endowed with some information which can be accessed by other agents forming links with them. Link formation is costly and links transmit information randomly. More precisely, agents in our model can form links and participate in a network by incurring a cost for each link, which may be interpreted in terms of time, money or effort. The cost of establishing a link is incurred only by the agent who initiates it, and the initiating agent has access to the other agent's information with a certain probability. In addition, he has access to the information from all the links of the other agent. Thus each link can generate substantial positive externalities of a non-rival nature in the network. Moreover, the flow of benefits through a link occurs both ways. It can differ across agents, since the strength of ties varies across agents (although all links cost the same) so that links fail with possibly different probabilities. This reflects the fact that in reality, communication often embodies a degree of costly uncertainty. We frequently have to ask someone to reiterate what they tell us, explain it again and even seek second opinions.

Foreign immigrants often form such networks. When an immigrant lands on the shores of a foreign country he usually has a list of people from the home country to get in touch with. Once contacted, some compatriots are more helpful than others. Often a substantial information exchange takes place in this process, where the new arrival learns about the foreign country, while providing the established immigrants current information about the home country and an opportunity to indulge in nostalgia.

Bala and Goyal (2000a, b) suggest telephone calls as an example of such networks. Another example of this kind (especially of the star networks considered here) is a LISTSERVE or an e-mail system. Costs have to be incurred in setting up and joining such electronic networks, but being a part of the network does not automatically ensure access to the information of other agents. Member participation rates in an electronic network often vary, and messages may get lost as in the celebrated "e-mail game" (Rubinstein, (1989)).[1]

Motivated by these examples, we, like Bala and Goyal (2000a, b) develop a non-cooperative model of network formation which generalizes theirs. The non-cooperative game formulation of network formation models typifies one of three strands of literature of major concern to us. The two other strands that have recently emerged in the context of economics and game theory are differentiated by their use of cooperative game theory and the notion of pairwise stability, repectively. The early cooperative literature treats costs as a set of constraints on coalition formation (see for example, Myerson (1977), Kalai *et al.* (1978) and Gilles *et al.* (1994)). An excellent survey of that literature can be found in van den Nouweland (1993), and Borm, van den Nouweland and Tijs (1994). Aumann and Myerson (1988) were the first to incorporate both costs and benefits of coalition formation. This line of research has been extended by Slikker and van den Nouweland (2000).

[1]Like most of the network literature we shall preclude the possibility of harmful information, like nuisance phone calls. The same is assumed for intermediate agents or indirect links in a network who function as purveyors of information between other agents without incurring any disutility.

Jackson and Wolinsky (1996) introduced the concept of pairwise stability (known from the matching literature) as an equilibrium concept in models of network formation. This gave rise to a completely new strand of the literature focusing on the tension between stability and efficiency. Pairwise stability requires mutual consent of a pair of agents for link formation whereas links can be deleted unilaterally. Dutta and Muttuswami (1997) and Watts (2001) refine the Jackson-Wolinksy framework further by introducing other stability concepts and derive implementation results for these concepts. Johnson and Gilles (2000) introduce a geographic dimension to the Jackson-Wolinsky model through spatial costs of link formation. For a recent survey of this literature see Jackson (2003).

Several dynamic models using pairwise stability have been investigated as well, starting with Jackson and Watts (2002a). Jackson and Watts (2002b) and Goyal and Vega-Rodondo (1999) consider coordination games played on a network. The choice of partners in the game is endogenous and players are periodically allowed to add or sever links. Droste *et al.* (2000) also analyze coordination games played on a network — with spatial costs of link formation.

The non-cooperative version of network formation has first been developed in two papers by Bala and Goyal (2000a, b). In all cases, agents choose to form links on the basis of costs and a (deterministic or stochastic) flow of benefits that accrue from links. Bala and Goyal assume that a player can create a one-sided link with another player by making the appropriate investment. Their assumption differs fundamentally from the concept of pairwise stability since mutual consent of both players is no longer required for link formation. They further investigate the reliability issue in networks by allowing links to fail independently of each other with a certain probability. Links are deterministic in Bala and Goyal (2000a). They are random, with identical probabilities of failure for all established links, in Bala and Goyal (2000b). Thus both their models deal with homogeneous agents. The corresponding static equilibrium outcomes are called Nash networks.[2]

Our model also belongs to the non-cooperative tradition and is a generalization of Bala and Goyal (2000b). We introduce agent heterogeneity by allowing for the probability of link failure (or success) to differ across

[2] They also identify strict Nash networks and study the formation of Nash networks in a modified version of best-response dynamics.

links. This distinctive feature reflects the nature of the transmission technology or the quality of information. The generalization provides a richer model in terms of answering theoretical as well as practical questions: connectivity and super-connectivity, selection of central agents in star networks, efficiency, and Pareto-optimality. Besides imparting greater realism to the model, the introduction of heterogeneous agents allows us to check the robustness of the conclusions obtained in Bala and Goyal (2000b). Whereas their findings still hold under certain conditions, heterogeneity gives rise to a greater variety of equilibrium outcomes, tends to alter results significantly and even generates some of the results of their deterministic model.

Bala and Goyal show for both their models that Nash networks must be either connected or empty. With heterogeneous agents, this proves true only when the probabilities of success are not very different from each other. Another central finding of Bala and Goyal is that compared to information decay imperfect reliability has very different effects on network formation. With information decay, minimally connected networks (notably the star) are Nash for a wide range of cost and decay parameters, independently of the size of society. In contrast, with imperfect reliability and small link formation costs, minimally connected networks tend to be replaced by super-connected networks (connected networks with redundant links) as the player set increases. However, with agent heterogeneity neither connectedness nor super-connectedness need arise asymptotically. Furthermore, in order for star networks to be Nash, probabilities must lie in a certain range (depending on costs) as in Bala and Goyal's setting. But we find that as a rule, they have to satisfy additional conditions. In particular, it never pays in the Bala and Goyal framework to connect to the center of the star indirectly. In our context, however, such a connection might be beneficial and further conditions on probabilities are required to prevent these connections. Interestingly enough, heterogeneity helps resolve a particular ambiguity associated with the homogeneous model: Owing to the additional equilibrium conditions, the coordination problem inherent in selecting the central agent of a star is mitigated to a certain degree.

We also investigate efficiency issues and find that Nash networks may be nested and Pareto-ranked. We demonstrate by example that inefficient Nash networks can be Pareto-optimal. A further example shows that Nash networks do not always exist with non-uniform link success probabilities. Criticisms of the non-cooperative approach to network formation are addressed as well. We extend the model to allow for duplication of links and to analyze Nash networks with incomplete information. Regarding the first extension, it turns out that double links enable the formation of some networks that could not occur in the single links model. In the second extension, we find that redundant links will be established when the agents beliefs about the probabilities of the indirect links are lower than the actual probabilities. Finally, the implications of mutual consent and endogenous success probabilities for Nash networks are discussed and explored.

The network literature almost completely lacks models with heterogeneous agents, with the notable exception of Johnson and Gilles (2000) and Droste *et al.* (2000) who introduce spatial heterogeneity of agents and obtain results substantially different from both static and dynamic versions of homogeneous-agent pairwise stability models. Their model and ours differ in two respects: the kind of agent heterogeneity and the equilibrium concept. They follow Jackson and Wolinsky (1996) and use *pairwise stability* as the equilibrium concept. We analyze Nash networks. In a recent paper, Galeotti and Goyal (2002) also incorporate heterogeneity in the context of Nash networks. In their model different links may have different costs, and the benefits of obtaining information from other players also varies across agents. They find that equilibrium networks continue to have some of the key features identified by Bala and Goyal (2000a), i.e., centrality, center-sponsorship and short network diameter occur in equilibrium despite the presence of heterogeneous players.[3] These properties depend on the fact that links are fully reliable.

In Section 2, we introduce the basic notation and terminology used throughout the paper. In Section 3, we present some general results and observations on Nash networks. Alternative formulations of the model are considered in Section 4. Section 5 concludes. Section 6 contains proofs and derivations.

## 2 The Model

Let $N = \{1, \ldots, n\}$ denote the set of agents, with generic members $i$ and $j$. For ordered pairs $(i, j) \in N \times N$, the shorthand notation $ij$ is used. For non-ordered pairs $\{i, j\}$, the notation $[ij]$ is used. The symbol $\subset$ for set inclusion permits equality. We assume throughout that $n \geq 3$. Each agent

[3] However, some novel and somewhat unexpected Nash network architectures are observed, for instance collections of stars.

has some information of value to the other agents. An agent can get access to more information by forming links with other agents. Agents form their links simultaneously. The formation of links is costly. Each link denotes a connection between a pair of agents which is not fully reliable. It may fail to transmit information with a positive probability that can differ across links.

Each agent's strategy is a vector $g_i = (g_{i1}, \ldots, g_{ii-1}, g_{ii+1}, \ldots, g_{in})$ where $i \in N$ and $g_{ij} \in \{0, 1\}$ for each $j \in N \backslash \{i\}$. The value $g_{ij} = 1$ means that agents $i$ and $j$ have a link initiated by $i$ whereas $g_{ij} = 0$ means that agent $i$ does not initiate the link. This does not preclude the possibility of agent $j$ initiating a link with $i$. A link between agents $i$ and $j$ potentially allows for **two-way (symmetric) flow of information**. The set of all pure strategies of agent $i$ is denoted by $\mathcal{G}_i$. We focus only on pure strategies in this paper. Given that agent $i$ has the option of forming or not forming a link with each of the remaining $n-1$ agents, the number of strategies available to agent $i$ is $|\mathcal{G}_i| = 2^{n-1}$. The strategy space of all agents is given by $\mathcal{G} = \mathcal{G}_1 \times \cdots \times \mathcal{G}_n$. A strategy profile $g = (g_1, \ldots, g_n)$ can be represented as a **directed graph** or **network**. Notice that there is a one-to-one correspondence between the set of all directed networks with $n$ vertices or nodes and the set of strategies $\mathcal{G}$. The link $g_{ij}$ will be represented pictorially by an edge starting at $j$ with the arrowhead pointing towards $i$ to indicate that agent $i$ has initiated the link. The reader may refer to Figure 2 below where agents 1 and 2 establish the links with agent 6 and agents 3 and 4 establish the links with agent 7. Consequently, the cost of forming these links are borne by agents 1, 2, 3 and 4 and the arrowhead always points towards the agent who pays for the link.

To describe information flows in the network, let for $i \in N$ and $g \in \mathcal{G}$, $\mu_i^d(g_i) = |\{k \in N : g_{ik} = 1\}|$ denote the number of links in $g$ initiated by $i$ which is used in the determination of $i$'s costs. Next we define the closure of $g$ which is instrumental for computing benefits, since we are concerned with the symmetric, two-way flow of benefits. Pictorially the closure of a network is equivalent to replacing each directed edge of $g$ by a non-directed one.

**Definition 1** *The* ***closure*** *of $g$ is a non-directed network denoted by $h = cl(g)$ and defined as $cl(g) = \{ij \in N \times N : i \neq j$ and $g_{ij} = 1$ or $g_{ji} = 1\}$.*

**Benefits.** The benefits from network $g$ are derived from its closure $h = cl(g)$. For two agents $i \neq j$, the non-ordered pair $[ij]$ represents the undirected or both-way link between $i$ and $j$, i.e. the simultaneous occurrence of $ij$ and $ji$. If $h_{ij} = h_{ij} = 1$, then $[ij]$ succeeds with probability

$p_{ij} \in (0, 1)$ and fails with probability $1 - p_{ij}$ where $p_{ij}$ is not necessarily equal to $p_{ik}$ for $j \neq k$. It is assumed, however, that $p_{ij} = p_{ji}$. Furthermore, the success or failure of direct links between different pairs of agents are assumed to be independent events. Thus, $h$ may be regarded as a random network with possibly different probabilities of realization for different edges. We call a non-directed network $h'$ a realization of $h$ (denoted by $h' \subset h$) if it satisfies $h'_{ij} \leq h_{ij}$ for all $i, j$ with $i \neq j$. The notation $[ij] \in h'$ signifies that the undirected link $[ij]$ belongs to $h'$, that is $h'_{ij} = h'_{ji} = 1$.

At this point the concept of a path (in $h'$) between two agents proves useful.

**Definition 2** *For $h' \subset h$, a **path** of length $m$ from an agent $i$ to a different agent $j$ is a finite sequence $i_0, i_1, \ldots, i_m$ of pairwise distinct agents such that $i_0 = i$, $i_m = j$, and $h'_{i_k i_{k+1}} = 1$ for $k = 0, \ldots, m - 1$. We say that player $i$ **observes** player $j$ in the realization $h'$, if there exists a path from $i$ to $j$ in $h'$.*

Invoking the assumption of independence, the probability of the network $h'$ being realized given $h$ is

$$\lambda(h' \mid h) = \prod_{[ij] \in h'} p_{ij} \prod_{[ij] \notin h'} (1 - p_{ij}).$$

Let $\mu_i(h')$ be the number of players that agent $i$ observes in the realization $h'$, i.e. the number of players to whom $i$ is directly or indirectly linked in $h'$. Each observed agent in a realization yields a benefit $V > 0$ to agent $i$. Without loss of generality assume that $V = 1$.

Given the strategy tuple $g$ agent $i$'s expected benefit from the random network $h$ is given by the following benefit function $B_i(h)$:

$$B_i(h) = \sum_{h' \subset h} \lambda(h' \mid h) \mu_i(h')$$

where $h = cl(g)$. The probability that network $h'$ is realized is $\lambda(h' \mid h)$, in which case agent $i$ gets access to the information of $\mu_i(h')$ agents in total. Note that the benefit function is clearly non-decreasing in the number of links for all the agents.

**Payoffs.** We assume that each link formed by agent $i$ costs $c > 0$. Agent $i$'s expected payoff from the strategy tuple $g$ is

$$\Pi_i(g) = B_i(cl(g)) - \mu_i^d(g_i)c. \tag{1}$$

Given a network $g \in \mathcal{G}$, let $g_{-i}$ denote the network that remains when all of agent $i$'s links have been removed. Clearly $g = g_i \oplus g_{-i}$ where the symbol

$\oplus$ indicates that $g$ is formed by the union of links in $g_i$ and $g_{-i}$.

**Definition 3** *A strategy $g_i$ is said to be a **best response** of agent $i$ to $g_{-i}$ if*

$$\Pi_i(g_i \oplus g_{-i}) \geq \Pi_i(g_i' \oplus g_{-i}) \text{ for all } g_i' \in \mathcal{G}_i.$$

Let $BR_i(g_{-i})$ denote the set of agent $i$'s best responses to $g_{-i}$. A network $g = (g_1, \ldots, g_n)$ is said to be a **Nash network** if $g_i \in BR_i(g_{-i})$ for each $i$, i.e., agents are playing a Nash equilibrium. A strict Nash network is one where agents are playing strict best responses.

Agent $i$'s benefit from the direct link $ij$ to agent $j$ is at most $p_{ij}(n-1)$. Set $p_0 = p_0(c, n) = c \cdot (n-1)^{-1}$. If $p_{ij} < p_0$, it never benefits agent $i$ to initiate a link from $i$ to $j$, no matter how reliably agent $j$ is linked to other agents and, therefore, $g_{ij} = 0$ in any Nash equilibrium $g$.

We now introduce some additional definitions which are of a more graph-theoretic nature. A network $g$ is said to be **connected** if there is a path in $h = cl(g)$, between any two agents $i$ and $j$. A connected network $g$ is said to be **super-connected**, if there exist links after whose deletion the network is still connected.[4] A connected network $g$ is **minimally connected**, if it is no longer connected after the deletion of *any* link. A network $g$ is called **complete**, if all links exist in $cl(g)$. A network with no links is called an **empty network**.

**Definition 4** *A set $C \subset N$ is called a **component** of $g$ if there exists a path in $cl(g)$ between any two agents $i$ and $j$ in $C$ and there is no strict superset $C'$ of $C$ for which this holds true.*

The commonly used welfare measure is defined as the sum of the payoffs of all the agents. Formally, let $W : \mathcal{G} \to \mathcal{R}$ be defined as

$$W(g) = \sum_{i=1}^{n} \Pi_i(g) \text{ for } g \in \mathcal{G}.$$

**Definition 5** *A network $g$ is **efficient** if $W(g) \geq W(g')$ for all $g' \in \mathcal{G}$.*

[4] Bala and Goyal (2000b) call a network super-connected if it remains connected after the deletion of an arbitrary single link.

An efficient network is one that maximizes the total value of information available to all agents net of the aggregate costs of forming the links. The definition of (strong) Pareto-optimality is the usual one: A network $g$ is **Pareto-optimal**, if there does not exist another network $g'$ such that $\Pi_i(g') \geq \Pi_i(g)$ for all $i$ and $\Pi_i(g') > \Pi_i(g)$ for some $i$. Obviously, every efficient network is Pareto-optimal. However, we will show that not every Pareto-optimal network is efficient. In fact, we provide an example of a Pareto-optimal Nash network which is inefficient, while the unique efficient network is not Nash.

We finally introduce the notion of an essential network. A network $g \in \mathcal{G}$ is **essential** if $g_{ij} = 1$ implies $g_{ji} = 0$. Note that if $c > 0$ and $g \in \mathcal{G}$ is a Nash network or an efficient network, then it must be essential. This follows from the fact that the information flow is two-way and independent of which agent invests in forming the link, that is $h_{ij} = \max\{g_{ij}, g_{ji}\}$. If $g_{ij} = 1$, then by the definition of $\Pi_j$ agent $j$ pays an additional cost $c$ for setting $g_{ji} = 1$, while neither he nor anyone else gets any benefit from it. Hence if $g$ is not essential it cannot be Nash or efficient.

# 3 Nash Networks

In this section we look at Nash networks. We begin with an analysis of connectedness and redundancy in Nash networks. Then we identify conditions under which complete networks and the empty network, respectively, will be Nash. This is followed by a subsection that covers the popular star networks. We also discuss efficiency issues by means of examples and existence by way of a counter-example.

## 3.1 Connectivity and Super-Connectivity

With homogeneous agents, Nash networks are either connected or empty (Bala and Goyal (2000b)). With heterogeneous agents, this dichotomy does not always hold. The proposition below identifies conditions under which Nash networks will exhibit this property.

PROPOSITION 1: *If $p_{ij} \geq \frac{1}{1+c/n^2} p_{mk}$ for any $i \neq j$ and $m \neq k$, then every Nash network is either empty or connected.*

**Proof**: Consider a Nash network $g$. Suppose $g$ is neither empty nor connected. Then there exist three agents $i$, $j$, and $k$ such that $i$ and $j$

belong to one connected component of $cl(g)$, $C_1$ and $k$ belongs to a different connected component of $cl(g)$, $C_2$. Then $g_{ij} = 1$ or $g_{ji} = 1$, whereas $g_{mk} = g_{km} = 0$ for all $m \in C_1$. Without loss of generality assume $g_{ij} = 1$. Then the incremental benefit to $i$ of having the link from $i$ to $j$ is $b_1 \geq c$. Let $g'$ denote the network which one obtains, if in $g$ all direct links with $i$ as a vertex are severed. The incremental expected benefit to $i$ of forming the link $ij$ in $g'$ is $b_2 \geq b_1 \geq c$ and can be written as $b_2 = p_{ij}(1 + V_j)$ where $V_j$ is $j$'s expected benefit from all the links $j$ has in addition to $ij$.

Now consider a link from $k$ to $j$, given $g' \oplus g_{ij}$. This link is worth $b_3 = p_{kj}(p_{ij} + 1 + V_j)$ to $k$. A link from $k$ to $j$, given $g$, is worth $b_4 \geq b_3$ to $k$. We claim that $b_3 > b_2$, i.e.,

$$p_{kj} > p_{ij} \frac{1 + V_j}{1 + V_j + p_{ij}}$$

Since $g$ is Nash and $g_{ij} = 1$, we know $p_{ij} \geq p_0 > c/n$. By assumption, $p_{kj} \geq \frac{1}{1+c/n^2} p_{ij}$. Therefore,

$$p_{kj} > \frac{1}{1 + p_{ij}/n} p_{ij} = p_{ij} \frac{1 + n - 1}{1 + n - 1 + p_{ij}} \geq p_{ij} \frac{1 + V_j}{1 + V_j + p_{ij}}$$

where we use the fact that $V_j$ is bounded above by $n - 1$. This shows the claim that $b_4 \geq b_3 > b_2 \geq b_1 \geq c$. Initiating the link $kj$ is better for $k$ than not initiating it, contradicting that $g$ is Nash. Hence to the contrary, $g$ has to be either empty or connected. ■

This result means that if the probabilities are not too widely dispersed, then the empty versus connected dichotomy still holds. If, however, the probabilities are widely dispersed, then a host of possibilities can arise and a single dichotomous characterization is no longer adequate.

Bala and Goyal (2000b) further show that with homogeneous agents and imperfect reliability, Nash networks become super-connected as the size of the society increases. This result warrants several qualifications. The first one concerns an obvious trade-off even in the case of homogeneous agents. While it is correct that for any given probability of success $p > 0$, super-connectivity obtains asymptotically, the minimum number of players it takes to get super-connectivity goes to infinity as $p$ goes to zero. Let $n^*$ be any number of agents. If $p < p_0(c, n^*)$, then it takes at least $n^* + 1$ agents to obtain even a connected Nash network.

Secondly, in our model with heterogeneous agents, asymptotic connectivity need no longer obtain, eliminating any scope for super-connectivity. Consider an infinite sequence of agents $i = 1, 2, \ldots, n, \ldots$ and a sequence of probabilities $p_2, p_3, \ldots$ such that $p_{ij} = p_{ji} = p_j$ for $i < j$. Then the sequence $p_k, k \geq 2$, can be constructed in such a way that the empty network is the only Nash network for any agent set $I_n = \{1, \ldots, n\}$, $n \geq 2$. Of course, with heterogeneous agents, asymptotic super-connectivity obtains, if there exist $q_0$ and $q_1$ such that $0 < q_0 \leq p_{ij} \leq q_1 < 1$ for all $ij$. The argument for homogeneous agents can easily be adapted to this case.

Finally, the lack of a common positive lower bound for the success probabilities does not necessarily rule out asymptotic super-connectivity, provided the probabilities do not drop too fast. A positive example is given by $c = 1$ and $p_{ij} = p_{ji} = p_j = j^{-1/2}$ for $i < j$. Basically, the argument developed for homogeneous agents can be applied here, too. This follows from the fact that for $1 < m < n$,

$$\sum_{i=m}^{n} p_{1i} > \int_{m}^{n+1} s^{-1/2} ds = [2s^{1/2}]_m^{n+1} = 2((n+1)^{1/2} - m^{1/2}).$$

Furthermore, with heterogeneous agents, other possibilities exist as well. For instance, super-connectivity may be established at some point, but connectivity may break down when further agents are added and reemerge later, etc. Or possibly several connected components persist with super-connectivity within each component. Thus the Bala and Goyal result is altered significantly in our model.

### 3.2 The Polar Cases

The next proposition identifies conditions under which the complete network and the empty network are Nash. Let $P = [p_{ij}]$ denote the matrix of link success probabilities for all agents $(i, j) \in N \times N$, where $p_{ij} \in (0, 1)$.

Proposition 2*: For any $P$, there exists $c(P) > 0$ such that each essential complete network is (strict) Nash for all $c \in (0, c(P))$. The empty network is strict Nash for $c > \max\{p_{ij}\}$.*

**Proof**: Let $g = g_i \oplus g_{-i}$ be any essential complete network. Consider an arbitrary agent $i$ with one or more links in his strategy $g_i$. Let $\mathcal{G}'_i = \{g'_i \in \mathcal{G}_i : g'_{ij} \leq g_{ij}$ for all $j \neq i\}$. Clearly, if $c = 0$ then for agent $i$, $g_i$ is a strict best response in $\mathcal{G}'_i$ against $g_{-i}$. By continuity, there exists $c_i(P, g_{-i}) > 0$ so that $g_i$ is a strict best response in $\mathcal{G}_i'$ against $g_{-i}$ for all $c \in (0, c_i(P, g_{-i}))$. Suppose $c \in (0, c_i(P, g_{-i}))$. If $g_i^* \in \mathcal{G}_i \backslash \mathcal{G}_i'$, then $g_{ij}^* = g_{ji} = 1$ for some $j \neq i$ and there exists a better response $g_i' \in \mathcal{G}_i'$ without redundant costly links. Since $g_i$ is at least as good a response as $g_i'$, it is also a better response than $g_i^*$. Hence for $c \in (0, c_i(P, g_{-i}))$, $g_i$ is a strict best response in $\mathcal{G}_i$ against $g_{-i}$. Now let $c(P)$ be the minimum of $c_i(P, g_{-i})$ over all conceivable combinations of $i$ and $g_{-i}$. The first part of the claim follows from this.

For the second part, if $c > \max\{p_{ij}\}$ and no other agent forms a link, then it will not be worthwhile for agent $i$ to form a link. Hence the empty network is strict Nash as asserted. ■

### 3.3 Star Networks

Star networks are among the most widely studied network architectures. They are characterized by one agent who is at the center of the network and the property that the other players can only access each other through the central agent. There are three possible types of star networks. The inward pointing (center-sponsored) star has one central agent who establishes links to all other agents and incurs the cost of the entire network. An outward pointing (periphery-sponsored) star has a central agent with whom all the other $n-1$ players form links. A mixed star is a combination of the inward and outward pointing stars. Here we will focus on the periphery-sponsored star and the proofs provided below can be easily adapted to the other types of stars.

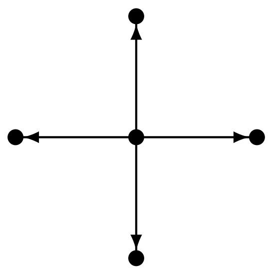

FIGURE 1: Outward Pointing (Periphery-Sposored) Star Network

While the method of computing Nash networks does not change with the introduction of heterogeneous agents, the process of identifying the different parameter ranges for Nash networks is now considerably more complicated. We will next establish two claims about Nash networks to illustrate the complex nature of the situation with agent heterogeneity. Without loss of generality we will assume that player $n$ is the central agent in the star. Define $M$ to be the set of all the agents except $n$ or $M = N\backslash\{n\}$ and let $K_m = M\backslash\{m\}$ be the set $M$ without agent $m$. Also let $J_k = K_m\backslash\{k\}$ denote a set $K_m$ without agent $k$ and $\Sigma_k = \sum_{j\in J_k} p_{jn}$. We shall use the following **star condition**:

For all $m \in M$, $k \in K_m$: Either $p_{mn} > p_{mk}$ or $[p_{mn} < p_{mk}$, $p_{mn} > p_{mk}p_{kn}$, and $(p_{mn} - p_{mk}p_{kn})\Sigma_k > (p_{mk} - p_{mn}) + p_{kn}(p_{mk} - p_{mn})]$.

PROPOSITION 3: *Given $c \in (0,1)$, there exists a threshold probability $\delta \in (0,1)$ such that the outward pointing star is Nash if :*

1. $p_{ij} \in (\delta, 1)$ for all pairs $ij$.
2. *The star condition holds.*

**Proof:** Consider the outward pointing star with agent $n$ as the central agent. Choose the threshold probability $\delta \in (c,1)$ to satisfy the inequality

$$\max_{m\in M}\left[(1-p_{nm}) + \left((n-2) - p_{nm}\sum_{k\in K_m} p_{nk}\right)\right] < c \qquad (2)$$

if $p_{ij} \in (\delta, 1)$ for all $ij$. Next we identify the conditions under which no player wants to deviate. We know that $n$ has no links to sever, and does not want to add a link since $g_{mn} = 1$ for all $m \in M$ and the flow of benefits is two-way. Now consider an agent $m \neq n$ who might wish to sever the link with $n$ and instead link with some other $k \in K_m$. Player $m$'s payoff from the outward pointing star is $\Pi_m(g^{ot}) = p_{mn} + p_{mn}\sum_{k\in K_m} p_{kn} - c$. His payoff from deviating and forming the new link is $\Pi_m(g^{ot} - g_{mn} + g_{mk}) = p_{mk} + p_{mk}p_{kn} + p_{mk}p_{kn}\Sigma_k p_{jn} - c$. We get $\Pi_i(g^{ot}) - \Pi_i(g^{ot} - g_{in} + g_{ik}) = (p_{mn} - p_{mk}) + p_{kn}(p_{mn} - p_{mk}) + (p_{mn} - p_{mk}p_{kn})\Sigma_k$. This is clearly positive when $p_{mn} > p_{mk}$ for all $m \in M$, i.e., when every non-central agent $(i \neq n)$ has her best link with the central agent .

However, when the inequality is reversed, we need $p_{mn} > p_{mk}p_{kn}$ and $(p_{mn} - p_{mk}p_{kn})\Sigma_k > (p_{mk} - p_{mn}) + p_{kn}(p_{mk} - p_{mn})$, i.e., agent $k$'s link with $n$ is so weak that it is not worthwhile for $m$ to form this link. Essentially, the difference between the benefits from accessing agents $j \in J_k$ through $n$ instead of the indirect link through $k$ in this case should exceed net benefits from agents $n$ and $k$ when agent $m$ establishes a link with $k$ instead of the central agent. Note that player $m$ can only sever one link in an outward pointing star and hence we need not consider any more instances of link substitution by player $m$.

Next we need to check that no agent wants to add an extra link. This means that no $m \in M$ wants to form a link with any $k \in K_m$. Note that payoffs with this additional link are bounded above by $(n-1) - 2c$. Taking the difference between $\Pi_m(g^{ot} + g_{mk})$ and $\Pi_m(g^{ot})$ we get $[(1 - p_{mn}) + (1 - p_{1n}p_{mn}) + \cdots + (1 - p_{m-1n}p_{mn}) + (1 - p_{m+1n}p_{mn}) + \cdots + (1 - p_{n-1n}p_{mn})] < c$ as the condition that the additional link is lowering $m$'s payoff. Verifying that this is satisfied for all $m \in M$, gives us $\max_{m \in M}[(1 - p_{mn}) + (1 - p_{1n}p_{mn}) + \cdots + (1 - p_{m-1n}p_{mn}) + (1 - p_{m+1n}p_{mn}) + \cdots + (1 - p_{n-1n}p_{mn})] < c$, which is equivalent to (**2**). Since we use the upper bound on the payoffs to show that it is not worthwhile to add even one extra link by any player $m \in M$, this obviates the need to check that a player may want to add more than one link. ■

Compared to the Bala and Goyal framework, the introduction of heterogeneous agents alters the situation significantly. While part of the difference involves more complex conditions for establishing any star network, heterogeneity comes with its own reward. A different probability for the success of each link resolves the coordination problem implicit in the Bala and Goyal framework. For a constant probability of success, once one identifies conditions under which a given star network will be Nash, the role of the central agent can be assigned to any player. With heterogeneous agents, however, there are some natural candidates for the central agent. The agent who has the least benefit net of costs from a single link, is the natural choice for the central agent in the outward pointing star. There are some other differences from the Bala and Goyal framework. Notice that the determination of $\delta$ involves probabilities of all other links, making it quite complicated. Further, the benefits from deviation are also altered now. In the Bala and Goyal framework, no agent in the outward pointing star will ever deviate by severing a link with the central agent. In our model, links to the central agent will be severed unless the probabilities in the relevant range satisfy some additional conditions.

Note that in our framework the inward pointing star is Nash in the above specified range of costs if the central agent's worst link yields higher benefits than $c$ and (**2**) is satisfied. Clearly, the role of the central agent for this star can be assigned to the agent whose payoff net of costs from forming the $(n-1)$ links is the highest. The mixed star can be supported as Nash when conditions required by the inward and the outward pointing star are satisfied for the relevant agents.

Next let us consider the case where $c > 1$. Here $c > p_{ij}$ for all links $g_{ij}$. We provide conditions under which the outward pointing star is Nash.

PROPOSITION 4: *Given $c \in (1, n-1)$ there exists a threshold probability $\delta < 1$ such that for $p_{ij} \in (\delta, 1)$ the outward pointing star is Nash.*

**Proof:** Let agent $n$ be the center with whom all the other players establish links. Since $c \in (1, n-1)$ we can choose $\delta \in (0,1)$ such that if $p_{ij} \in (\delta, 1)$ for all $ij$, then (**2**) holds and $\min_{m \in M}[p_{mn}(1 + \sum_{k \in K_m} p_{kn})] > c$. Then no $m \in M$ wants to sever his link with $n$. The remainder of the proof is similar to the proof of Proposition 3. ∎

Once again it is possible to identify a natural candidate for the role of the central player. Also, note that the inward pointing and mixed star will never be Nash within this range of costs.

### 3.4 Efficiency Issues

Efficiency is a key issue in Jackson and Wolinsky (1996), Bala and Goyal (2000a,b), and Johnson and Gilles (2000). When costs are very high or very low, or when links are highly reliable, there is virtually no conflict between Nash networks and efficiency in the Bala and Goyal (2000b) framework. This observation still holds in our context. However, there is a conflict between Nash networks and efficiency for intermediate ranges of costs and link reliability, even with the same probability of link failure for all links. In particular, Nash networks may be under-connected relative to the social optimum as the subsequent example shows.

Let us add two important observations not made before. First, it is possible that Nash networks are nested and Pareto-ranked. Second, at least in our context, the following can coexist: a Nash network which is not efficient, but Pareto-optimal and a unique efficient network which is not Nash and does not weakly Pareto-dominate the Nash network. The first observation is supported by the following example: $c = 1$, $n = 4$ and $p_{ij} = 0.51$ for all $ij$. In this case, both the empty network and the outward pointing star with center 4 are Nash networks. The "outward pointing star" consisting of the links 14, 24 and 34 contains and strictly Pareto-dominates the empty network. Moreover, the empty network is under-connected. Our second observation is based on the following example, depicted in Figure 2.

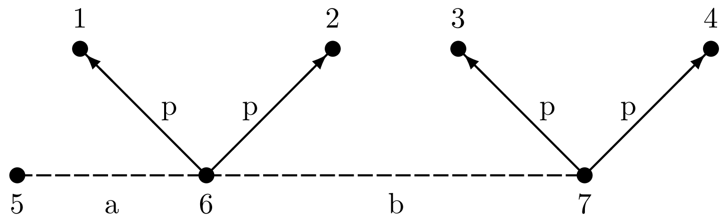


FIGURE 2: Inefficient and Pareto Nash Network

**Example 1**: $c = 1$, $n = 7$. $p_{16} = p_{26} = p_{37} = p_{47} = p = 0.6181$, $p_{56} = a = 0.2$, $p_{67} = b = 0.3$, and corresponding probabilities for the symmetric links. All other links have probabilities $p_{ij} < p_0$. Now $g$ given by $g_{16} = g_{26} = g_{37} = g_{47} = 1$ and $g_{ij} = 0$ otherwise is a Nash network. Indeed, $p$ is barely large enough to make this a Nash network. The critical value for $p$ satisfies $p(1+p) = 1$ with solution 0.6180.... But $g$ is not efficient. Linking also 5 with 6 and 6 with 7 provides the following added benefits where we use $2p = 1.2362$ and $1 + 2p = 2.2362$:

| | | | |
|---|---|---|---|
| For 1+2: | $1.2362 \cdot (a + b \cdot 2.2362)$ | $=$ | 1.07656 |
| For 3+4: | $1.2362 \cdot b \cdot (a + 2.2362)$ | $=$ | 0.90349 |
| For 5: | $a \cdot 2.2362 + ab \cdot 2.2362$ | $=$ | 0.58141 |
| For 6: | $a + b \cdot 2.2362$ | $=$ | 0.87086 |
| For 7: | $b \cdot (a + 2.2362)$ | $=$ | 0.73086 |
| **Total:** | | | 4.16318 |

Hence the total added benefit exceeds the cost of establishing these two additional links. The network thus created would be efficient. But neither 6 nor 7 benefits enough from the additional link between them to cover the cost of the link. Thus, the enlarged efficient network is not Nash. Since the rules of the game stipulate that one of the two agents assumes the entire cost of the new link, the enlarged efficient network cannot weakly Pareto-dominate $g$. In fact, $g$ is Pareto-optimal while inefficient. Reconciling efficiency and Pareto-optimality would require the possibility of cost sharing and side payments. ■

### 3.5 Existence of Nash Equilibria

The existence of Nash networks (or of pairwise stable networks) has not been systematically explored in the literature. Jackson and Watts (2002a) provide an example for non-existence of pairwise stable networks. Jackson (2003) shows existence of pairwise stable networks for several prominent allocation rules. Bala and Goyal (2000a) outline a constructive proof of the existence of Nash networks under perfect reliability. In addition, the literature contains assertions that for certain parameter ranges, the model admits Nash networks (or pairwise stable networks, respectively) with specific properties. If the various regions happen to cover the entire parameter space, then as a by-product, existence has been shown for the particular model. This is the case for $n = 3$ in the model of Bala and Goyal (2000b). If the various regions do not cover the entire parameter space, existence remains an open question for some parameter constellations. We going to show that when links have different success probabilities, a Nash network may not exist, i.e., heterogeneity can lead to non-existence of Nash equilibria.

**Example 2**: Let there be a total of 83 agents labelled $i = 0, 1, 2, 3, 301, \ldots, 363, 4, 401, \ldots, 415$. Set

$$\begin{array}{rcrcrcl} p &=& p_{12} &=& p_{21} &=& 0.4; \\ q &=& p_{23} &=& p_{32} &=& 0.01473; \\ r &=& p_{34} &=& p_{43} &=& 1/32; \\ s &=& p_{14} &=& p_{41} &=& 1/16; \\ t &=& p_{20} &=& p_{02} &=& 1. \end{array}$$

Further put

$$\begin{array}{rcrcll} p_{3j} &=& p_{j3} &=& 1 & \text{for } j = 301, \ldots, 363; \\ p_{4j} &=& p_{j4} &=& 1 & \text{for } j = 401, \ldots, 415; \\ p_{ij} &=& p_{ji} &=& 0 & \text{for all remaining } ij. \end{array}$$

Finally, choose $c = 0.95$. Then the following links will always be established:

$$\begin{array}{lcll} 02 & \text{or} & 20; & \\ 3j & \text{or} & j3 & \text{for } j = 301, \ldots, 363; \\ 4j & \text{or} & j4 & \text{for } j = 401, \ldots, 415. \end{array}$$

Obviously, none of the links $ij$ with $p_{ij} = 0$ will be established. Moreover, 1 will always establish the link 14, 4 will always establish the link 43, 2 will never establish the link 21 and 3 will never establish the link 32. Now the existence of a Nash network can be decided by assessing the benefits from links 12 and 23 to players 1 and 2, respectively, given that all other links have been established or not according to our foregoing account. We obtain:

- Without 23, player 1 strictly prefers not to establish 12.
- With 23, player 1 strictly prefers to establish 12.
- Without 12, the benefit to player 2 from link 23 is 0.95011 and establishing 23 is a strict best response.
- With 12, player 2's benefit from link 23 is reduced by $81pqrs = 0.00093$ (due to redundancies) and not establishing 23 is a strict best response.

Hence there are no mutual best responses regarding establishment of 12 and 23. Consequently, a Nash network does not exist. ■

To understand why the particular choice of $q$ has player 2 switch back and forth, replace $q$ by a $\widetilde{q}$ such that without 12, player 2 is indifferent between having and not having the link 23, i.e. $\widetilde{q} \cdot (64 + r \cdot 16 + rs) = c$. This yields $\widetilde{q} = 0.014728236$. Then with 12, player 2 would not want the link because of redundancies. By continuity, $q$ slightly larger than $\widetilde{q}$ produces the best response properties exhibited above.

# 4 Alternative Model Specifications

In this section we will consider four alternative specifications of our current model. The first variation introduces greater realism in the formation of networks by allowing agents to duplicate existing links, thereby raising link success probabilities. The second specification considers network formation under incomplete information. Here, each agent $i \in N$ is aware of the success probabilities $p_{ij}, i \neq j$ of her own links, but is ignorant of the probabilities of link successes of the other agents. Further, we discuss the implications for Nash networks when pairwise link formation requires the consent of the other agent. Finally, we examine Nash networks with endogenous link success probabilities.

## 4.1 An Alternative Formulation of Network Reliability

The payoff function in the previous section is based on the closure of the network implying that the links $g_{ij} = 1$ and $g_{ji} = 1$ are perfectly correlated.

Thus, no agent will ever duplicate a link if it already exists. An alternative way of modelling information flows would be to assume that the events $g_{ij} = 1$ and $g_{ji} = 1$ are independent. This captures the idea that if both agents involved in a relationship form the link, it will succeed with a higher probablity. Then a direct link between $i$ and $j$ is in effect with probability $r_{ij} = 1 - (1 - p_{ij})^2$ if $g_{ij} + g_{ji} = 2$, with probability $p_{ij}$ if $g_{ij} + g_{ji} = 1$, and zero probability if $g_{ij} + g_{ji} = 0$. Notice that $r_{ij} > p_{ij}$ for $p_{ij} \in (0, 1)$. Therefore, under certain circumstances, there exists an incentive for a double link between agents $i$ and $j$. This never occurs in the previous model since duplicating a link can only increase costs. We examine the implications of such double links while retaining the assumption that $p_{ij} = p_{ji}$. Also, the flow of benefits is still both ways. Potential new links can lead to new equilibria over a range of costs and probabilities. They may leave the original equilibrium unaltered or destroy it. The next example demonstrates several possibilities.

**Example 3**: Let $n = 3$. First consider the following probabilities and costs: $p_{12} = 0.8$, $p_{23} = 0.9$ and $c = 0.7$. All other link probabilities are zero or close to zero. Let $g^1 \equiv (g_{21} = g_{23} = 1)$. It is easy to verify that this simple center sponsored star is Nash since player 2 does not wish to break any links. Now suppose we allow for double links. It is easy to verify that the original center sponsored star remains Nash. With the current numerical specification, in any network involving double links, one of the agents contributing to a double link is better off breaking it. Next consider the case where $p_{12} = 0.5$, $p_{23} = 0.3$ and $c = 0.2$, with all other probabilities being zero. Once again when no double links are permitted the network $g^1$ is Nash. In contrast to the previous case, it is easy to verify that when double links are allowed, the equilibrium architecture is given by $g_{12} = g_{21} = 1 = g_{23} = g_{32}$. Finally, consider the case where $p_{12} = 0.5$, $p_{23} = 0.3$ and $c = 0.22$, with all other probabilities being zero. The center sponsored star is still Nash under the single link model. Once we allow for double links however, it is easy to verify that the Nash network is given by $g_{12} = g_{21} = g_{32} = 1$ with all other links being inactive. ■

The incremental reliability of a double link is given by $r_{ij} - p_{ij} = p_{ij} - p_{ij}^2$ which is maximized at $p_{ij} = 0.5$. Double links are not profitable when link costs exceed incremental benefits, which is most likely to occur when the probabilities are very high or low. This is exactly what happens with the first set of $p$ and $c$ parameters in the example. In the second numerical specification, players 1 and 3 gain from adding double links to the original network architecture, that is the incremental benefits provided by the double links strictly outweigh the costs, and link costs are sufficiently low so that 2 wants to keep both his links. In the last numerical specification, creation of the link 32 is beneficial for 3, but reduces the benefit of the prior link 23 for 2 from $p_{23} = 0.3$ to $r_{23} - p_{23} = 0.21$ which is less than its cost. Hence in response to the establishment of link 32, the prior link 23 is dropped. From the example, it should also be evident that this formulation can lead to super-connected networks of a different sort $-$ where agents may reinforce existing higher probability links instead of creating new links to players with whom they are not yet directly linked.

Observe that when double links are possible the earlier terminology of inward and outward pointing stars is not very helpful at times. For instance, using our old definitions it is not possible to describe the third equilibrium architecture in Example 2. We define a **degree two star** as a star network where every peripheral agent has a link with the central agent and the central agent has a link with each peripheral agent. This star combines the inward and outward pointing stars. Clearly, the definition of a mixed star also takes a new meaning now, since a mixed star can include both double links and single links that are initiated by either the central or a peripheral agent. Similarly, we can define a **degree two connected network** as a connected network where all links are double links. In a mixed connected network, both double and single links exist. Also, the **degree two complete network** is the complete network where every pair of agents has a double link between them. This allows us to obtain the analogue of Proposition 2. To begin with, a version of Proposition 1 also holds under this alternative formulation.

Remark 1 (Proposition 1′): *If $p_{ij} \geq \frac{1}{1+c/n^2} r_{mk}$ for any $i \neq j$ and $m \neq k$, then every Nash network is either empty or connected.*

The proof resembles the previous proof, while allowing for the possibility that there is a double link between $i$ and $j$. The condition on success probabilities implies for uniform success probability $p$ that $p \geq 1 - c/n^2$ which in turn implies $p - p^2 < p_0(n, c)$. This means that double links are not worth initiating and, consequently, connected Nash networks are essential although double links are allowed. Example 3 illustrates that for specific parameter constellations, Nash networks are connected with some or all links duplicated. Further, a version of Proposition 2 continues to hold. Let again $P = [p_{ij}]$ denote the matrix of link success probabilities for all agents

$(i, j) \in N \times N$ where $p_{ij} \in (0, 1)$.

REMARK 2 (Proposition 2′): *For any* $P$, *there exists* $c(P) > 0$ *such that the degree two complete network is (strict) Nash for all* $c \in (0, c(P))$. *The empty network is strict Nash for* $c > \max\{p_{ij}\}$.

With regard to the second part, the possibility of double links does not alter the fact that if $c > \max\{p_{ij}\}$ and no other agent forms a link, then it will not be worthwhile for agent $i$ to form a link. For the first part, the proof relies on a continuity argument and is similar to the proof of Proposition 2. For $n = 3$ and a uniform success probability $p \in [1/4, 1)$, one can show that if the degree two complete network is Nash, then the essential complete networks are not Nash. For instance, for $c = 0.25$ and $p = 0.71727$, the wheel with links 12, 23, 31 is Nash and, consequently, the degree two complete network cannot be Nash. In general, it is an open question whether other complete Nash networks can coexist with the degree two complete network. Example 3 offers three cases where the only Nash network is connected but not complete, where all, some, or none of the links are double links.

The consequences of the new formulation (effective double links) are now further explored by reexamining Proposition 3. The incentives for modifying links by deviating do not change under this formulation, i.e., the "star condition" of Proposition 3 is assumed to hold. The main impact of the double link model is on the threshold probability value $\delta$, altering the range of costs and probabilities under which the outward pointing star can be supported as Nash. Note that the payoff function used earlier for determining the payoff from an additional link gets around this issue by assuming that payoffs have an upper bound of $(n-1) - \alpha c$ where $\alpha$ denotes the number of links formed. In order to see how this new formulation will affect reliability we need to compute the precise value of the payoffs from additional links instead of using the upper bound. We denote the resulting new threshold value by $\widetilde{\delta}$. We find that a threshold of the form

$$\widetilde{\delta} = \max_{m \in M} \max_{i \neq m} \delta_i^m$$

will suffice. Each $\delta_i^m$ is a threshold value related to the specific link $mi$.

PROPOSITION 5: *Suppose that the links* $g_{ij} = 1$ *and* $g_{ji} = 1$ *are independent, and* $c \in (0, 1)$. *Then there exists a threshold probability* $\widetilde{\delta} \in (0, 1)$ *such that the outward pointing star is Nash if :*

1. $p_{ij} \in (\widetilde{\delta}, 1)$ for all pairs $ij$.

2. *The star condition holds.*

**Proof**: See Appendix. ■

In our previous formulation, $\delta$ can also be obtained as the maximum of link-specific thresholds. The latter tends to be lower if duplication has no benefits. Thus, in general $\widetilde{\delta} > \delta$. To see why this is the case let $p_{ij} = p$ for all $ij \in N \times N$. Then $p > \delta$ is equivalent to (i) $(1-p) + (1-(n-2)p^2) < c$. On the other hand, $p > \widetilde{\delta}$ is equivalent to (ii) $p(1-p^2) + (n-2)(1-p)p^2 < c$ plus (iii) $p(1-p) + (1-p)(n-2)p^2 < c$. Now $p = 1 - c$ satisfies (i) if $n \geq 2 + 1/(1-c)^2$, but violates (ii) and (iii) for sufficiently large $n$. Hence given $c$, one obtains $\widetilde{\delta} > \delta$ for sufficiently large $n$. Such is the case for $c = 1/2, n = 10$. Moreover, based on the example provided above, it is also obvious that a whole range of mixed stars may arise as equilibria.

## 4.2 Nash Networks under Incomplete Information

The previous sections have assumed that the agents are fully aware of all link success probabilities. However, this is not always a very realistic assumption. As an alternative, we now introduce incomplete information in the game. Each agent $i \in N$ has knowledge of the probability of success of all her direct links. However, she is not aware of the probability of success of indirect links, i.e., agent $i$ knows the value of $p_{ij}$, but is unaware of the value of $p_{jk}$, where $i \neq j, k$. The assumption that $p_{ij} = p_{ji}$ is still retained. We re-examine Proposition 3 for this specification.

In order to solve for the equilibrium networks, each agent $i$ must now have some beliefs about indirect links. We assume each agent postulates that, on average, every other agent's world is identical to her own. She assigns the average success value of all her own direct links to the indirect links, imparting a symmetry to the problem of indirect links. Thus, agent $i$ assigns a value of $p_i = \frac{1}{n-1} \sum_{i \neq m} p_{im}$ to all indirect links $p_{jk}$ for $i \neq j, k$. This has some immediate consequences for the payoff function. Consider some agent $m \in M$. This agent now believes that her payoff from the outward pointing star is given by $\Pi_m(g^{ot}) = p_{nm} + |K_m| \, p_{nm} p_m - c = p_{nm} + (n-2) p_{nm} p_m - c$, which is clearly different from her actual payoff. We obtain the **modified star condition** by replacing each term $p_{nk}$ with $p_m$. Let $\delta$ be the probability threshold of Proposition 3.

PROPOSITION 6*: Given each agent's beliefs about her indirect links, the outward pointing star is Nash if:*

1. $p_{ij} \in (\delta, 1)$ for all pairs $ij$.
2. *The modified star condition holds.*
3. $(n-2)(1-p_{nm}p_m) < ((n-2) - p_{nm}\sum_{k \in K_m} p_{nk})$ *for all* $m \in M$.

**Proof**: See Appendix. ■

This formulation provides us with some interesting insights about the role of the indirect links and the vulnerability of Nash networks. First observe that the set of conditions for the outward pointing star to be Nash are very similar to those in Proposition 3. However, there are some differences. For the purpose of comparison, let us assume that the actual probabilities satisfy the hypothesis of Proposition 3. It is possible that $1 - p_{nm} + (n-2)(1-p_{nm}p_m) > c > 1 - p_{nm} + ((n-2) - p_{nm}\sum_{k \in K_m} p_{nk})$, in which case agents will create new links destroying the star architecture. Consequently, the realized network yields lower payoffs than the star network. This is one of the crucial differences arising from this formulation. While the outward star is Nash under complete information, agents create additional links under incomplete information because of incorrect beliefs about indirect links. Thus, the introduction of incomplete information can easily lead to *network failure*, in the sense that the outcome is less efficient than it would be otherwise.

We now examine additional consequences of this formulation through an explicit example.

**Example 4:** Consider a network with $n = 6$. Suppose agents 1 to 4 are linked in a star formation with agent 4 being the central agent, i.e., $g_{14} = g_{24} = g_{34} = 1$. Further $g_{56} = 1$ and we will examine what happens to the link $g_{45}$ under complete and incomplete information. Let $c = 1/12$, $p_{14} = p_{24} = p_{34} = p = 4/10$, $p_{56} = r = 1/2$ and $p_{45} = q = 1/24$. The probabilities of all other links are assumed to be zero.

Under these objective probabilities it is easy to verify that $q(1 + r) < c$ and hence agent 4 will never initiate the link with agent 5. However, agent 5 will initiate this link since $q(1+3p) > c$. The resulting connected network is Nash since all other links yield no benefits.

Note that for our current formulation with incomplete information, $p_5 = \overline{r} = \frac{1}{5}(r + q)$ and $p_4 = \overline{p} = \frac{1}{5}(3p + q)$. Under these beliefs about the probabilities of the indirect links, agent 4 will never establish the link since $q(1+\overline{r}) < c$. Similarly, agent 5 will not establish the link since $q(1+3\overline{p}) < c$. With incomplete information the above disconnected network with $g_{45} = 0$ is a Nash network. ■

Thus incomplete information may destroy a crucial link and give rise to two connected components. In this section we demonstrate that incomplete information as modelled here can either lead to new links yielding lower payoffs or destroy crucial links in the network. There are other interesting alternatives to introduce incomplete information into a strategic model of network formation. Specifically, McBride (2002) considers a model with fully reliable links, but with incomplete information about the existence of certain indirect links and incomplete information about the benefits which accrue to a player via each of her direct links. He employs a generalized conjectural equilibrium concept.

### 4.3 Mutual Consent

This subsection explores the role of consent in Nash networks. In our setting and in much of the literature, it is assumed that agent $i$ does not need the consent of agent $j$ to initiate a link from $i$ to $j$. All it takes is that agent $i$ incurs the cost $c$. This could be construed as a drawback of the non-cooperative formulation. However, one might argue that when asked agent $j$ might give her permission anyway, since she will only benefit from an additional link that does not cost her anything.[5] Thus introducing an implicit consent requirement seems inconsequential, a descriptive improvement at best, a notational burden at worst. Yet Nash networks have another more serious weakness. Namely, it seems somewhat preposterous that agent $j$ should divulge all the information from her other links without even requiring her consent. For this reason, we now discuss the implications of a consent game. Formally, such a requirement can be accommodated by replacing each player's strategy set $\mathcal{G}_i$ by $\mathcal{G}_i \times \mathcal{G}_i$ with generic elements

[5] This argument is less compelling in the case of one-way information flow.

$(g_i, a_i) = (g_{i1}, \ldots, g_{ii-1}, g_{ii+1}, \ldots, g_{in}; a_{i1}, \ldots, a_{ii-1}, a_{ii+1}, \ldots, a_{in})$ where the second component, $a_i$, stands for $i$'s consent decisions. A link from $i$ to $j$ is initiated by mutual consent if and only if $g_{ij} = 1$ and $a_{ji} = 1$. Agents incur only the cost of links that are permitted. Denied links are absolutely costless.

Every graph $g$ that was a Nash network before is still a Nash network. But now there is room for mutual obstruction: $g_{ij} = 0$ is always a best response to $a_{ji} = 0$ and vice versa. Therefore, the empty network is always Nash under a mutual consent requirement. In addition, for any $N' \subseteq N$, the Nash networks with reduced player set $N'$ give rise to Nash networks of the network formation game requiring mutual consent with player set $N$, if one adds the agents in $N \setminus N'$ as isolated nodes. Furthermore, take any set of potential edges $E \subseteq N \times N$ and replace $p_{ij}$ by $q_{ij} < p_0$ for all $ij \in E$ in the original model. Any Nash network of the thus defined hypothetical game constitutes a Nash network of the network formation game requiring mutual consent.

But how robust are these new equilibria? Note that one could modify the mutual consent game by requiring that agents must incur the cost of all links they initiate, regardless of consent. Since agents are rational and have complete information, links that will be denied will never be initiated in equilibrium. Hence the Nash networks will be identical under this specification. However, all the new equilibria from the mutual consent game will be eliminated, if one imposes 2-player coalition-proofness or introduces conjectural variations of the kind that a player interested in initiating a link presumes the other's consent. A more serious issue is why two agents cannot split the cost in a Pareto-improving way when both would benefit from an additional link. Addressing endogenous cost sharing in a satisfactory way necessitates a radically different approach which is beyond the scope of the present generation of models.

The Jackson-Wolinsky "connections model" assumes exogenous cost sharing. In such a case, agent $i$ can have an incentive to reject a link from $j$ to $i$. More generally, one can consider a modification of the payoff function (**1**) that yields the expanded form

$$\Pi_i(g) = B_i(cl(g)) - \sum_j g_{ij} a_{ji} c - \sum_j g_{ji} a_{ij} c' \tag{3}$$

where $i$ incurs the cost or disutility $c'$, if he agrees to a link initiated by agent $j$. The quantity $c'$ can be interpreted as a composite cost which includes an explicit cost contribution towards the creation of a link $ji$ as well as a certain disutility (negative externality) that $i$ experiences when others contact him through this link.[6] The special case $c = c'$ is tantamount to equal cost sharing. While we leave the detailed analysis of reliability issues in this framework for future research, three observations can be made without further scrutiny. First, if giving one's consent is costly, Nash networks tend to be smaller, a finding echoed by Gilles and Sarangi (2003) for the full reliability case. This would still be true, if we allowed for the additional possibility that at an extra cost, agent $j$ can impose the link $ji$ against $i$'s objection. Second, the possibility of mutual obstruction persists under costly consent. Gilles and Sarangi (2003) show that for arbitrary network payoff functions, any equilibrium in a model with two-sided costs of link formation is also an equilibrium in a model with one-sided costs like the model developed in this paper. The reverse however is not true. It is easy to see that this observation holds for our model as well. Third, since the empty network is always Nash with a mutual consent requirement, existence is no longer an issue. In Example 2, a non-trivial equilibrium is possible as well, if mutual consent is required. Namely, $g_{12} = a_{21} = 0, g_{23} = a_{32} = 1$ are the crucial components of a Nash equilibrium where link 23 is formed and link 12 is not formed.

### 4.4 Endogenous Link Probabilities

In this subsection, we consider the possibility that the addition of a link renders all adjacent links less reliable. For conceivably, any given node might become less effective in responding to information requests via its direct links, if it gets accessed through one more direct link. In other words, the additional link causes a negative externality on the other links competing for access to the same node. Incorporating this particular feature into a model of network formation leads to endogenous failure probabilities. One of the consequences is that a complete network need no longer be Nash, even if links are costless. To illustrate this and other interesting possibilities, let us consider

[6] In a more refined version, one could differentiate the cost structure further and make the cost of accepting a specific link $ji$ depend on whether or not there is duplication, that is the reverse link $ij$ is initiated by $i$ and accepted by $j$.

**Example 5:** *Negative link externalities.*

Let $c = 0$. For $i \in N$ and $g \in \mathcal{G}$, set

$$n_i(g) = |\{k \in N \setminus \{i\} : g_{ik} = 1 \text{ or } g_{ki} = 1\}|,$$

the number of agents to whom $i$ has direct links in $g$. For any two agents $i$ and $j$ and any network $g$, let the endogenous probability of success of link $ij$ be given as

$$p_{ij}(g) = \begin{cases} \frac{1}{n_i(g)} \cdot \frac{1}{n_j(g)}, & \text{if } g_{ij} + g_{ji} > 0; \\ 0, & \text{if } g_{ij} + g_{ji} = 0. \end{cases}$$

First consider the case $n = 3$ and the wheel or circle $g$ with links 12, 23, and 31, an essential complete network where each link has success probability 1/4. Each player $i$ receives payoff $\Pi_i(g) = 19/32$. After severance of the link initiated by him, the two remaining links have each success probability 1/2 and $i$'s payoff becomes 3/4 or 24/32. This shows our claim that with endogenous success probabilities and zero or negligible costs, complete networks need no longer be Nash — in stark contrast to Proposition 2. Moreover, for $n \geq 4$, wheels with simple links, line networks with simple links, and stars are not Nash under the current assumptions. Regarding stars, a peripheral agent gains from initiating links to other peripheral agents in addition to the existing link to the central agent. Finally, the example exhibits non-empty Nash networks with very small connected components. It turns out that a network $g$ is Nash if each component $C$ either satisfies $|C| = 3$ and is incomplete (is not a wheel) or satisfies $|C| = 2$. ■

This somewhat extreme example clearly shows that the negative externality caused by additional links can affect the outcomes significantly. Next let us consider a more general model of endogenous success probabilities. It encompasses the case of capacity constraints where an agent or node $i$ cannot have or handle more than $L_i$ links.

We assume $P_{ij} = P_{ji} \in (0, 1]$ for $i \neq j$ and a non-increasing function $\alpha_i : \mathbb{N} \to [0, 1]$ with $\alpha_i(1) = 1$ for each agent. Then the endogenous success probabilities are given by

$$p_{ij}(g) = \begin{cases} \alpha_i(n_i(g))\alpha_j(n_j(g))P_{ij}, & \text{if } g_{ij} + g_{ji} > 0; \\ 0, & \text{if } g_{ij} + g_{ji} = 0. \end{cases}$$

In Example 5, $\alpha_i(z) = 1/z$. Another example is given by capacity constraints $L_1, \ldots, L_n$ where the $L_i$ are positive integers. For this case $\alpha_i(z) = 1$ for $z \leq L_i$ and $\alpha_i(z) = 0$ for $z > L_i$. The following obvious proposition confirms the conclusions drawn from Example 5.

Proposition 7: *If $(n-1)\alpha_i(n-1) < c^{1/2}$ for all $i$, then no star network can be Nash.*

In the case of capacity constraints, suppose $c \in (0,1)$ and $P_{ij} = p \in (0,1)$. Then the existence of star Nash networks depends on the severity of the capacity constraints.

Proposition 8: *There exists $p(c) \in (c,1)$ such that for $p \in (p(c),1)$:*

(i) *If for all $i$, $L_i < n-1$, then no star network is Nash.*

(ii) *If for some $i$, $L_i \geq n-1$, then all stars with center $i$ are Nash.*

**Proof:** By Proposition 3.2(b) of Bala and Goyal (2000b), there exists $p(c) \in (c,1)$ such for $p \in (p(c),1)$, all star networks are Nash in the absence of capacity constraints. If $L_i < n-1$ and $i$ is the center of a star, then $p_{ij} = 0$ for all $j \neq i$ and an agent is better off severing a link to or from $i$. Hence (i). If $L_i \geq n-1$ and $i$ is the center of a star, then $p_{ij} = p$ for all $j \neq i$ and the star remains Nash after the imposition of the capacity constraints. Hence (ii). ■

The foregoing example and the two propositions suggest that negative link externalities expressed in terms of endogenous success probabilities tend to destabilize some of the prominent network architectures. However, novel and equally interesting Nash networks may arise as the following example illustrates.

**Example 6.** Let $n = 4$, $c = 0.35$, $P_{1j} = P_{j1} = 0.8$ for $j \neq 1$, and $P_{ij} = 0.5$ for all other links. With exogenous probabilities $p_{ij} = P_{ij}$, all stars with center 1 are Nash, whereas the wheel with links $12, 23, 34, 41$ is not Nash because 1 gains from establishing the extra diagonal link 13. Now assume endogenous probabilities and $\alpha_i(1) = \alpha_i(2) = 1$ for all $i$. Then for sufficiently small values of $\alpha_i(3)$ and $\alpha_i(4)$, the stars cease to be Nash and the above wheel becomes Nash. This shows that network externalities may destabilize certain prominent network architectures and help support others. ■

The approach taken here has provided several immediate and important insights. The asymptotic behavior of the network (as $n$ increases) crucially depends, among things, on the properties of the functions $(n-1)\alpha_i(n-1)$. Again, super-connectivity may or may not occur. Under certain circumstances, connected multi-hub systems can emerge.

The present approach can be generalized in several ways. The multiplicative form $\alpha_i\alpha_j$ incorporates both some degree of substitutability and some degree of complementarity between nodes. The additive form $\alpha_i + \alpha_j$, after suitable normalization, would reflect perfect substitutability and the form $\max\{\alpha_i, \alpha_j\}$ perfect complementarity. Hence a richer model of endogenous probabilities remains to be investigated.

Goyal and Joshi (2003), in a model with full reliability, but more general utility functions, view the role of externalities in link formation from a different angle. They focus on how the marginal benefit of an extra link is affected by the number of links of the agent or the number of links of other agents. In our context, with imperfect reliability and exogenous success probabilities, an increase of the number of direct links to or from an agent typically decreases and never increases the agent's marginal benefit from a direct link. The effects of additional links by other agents can be of either sign, both with exogenous and endogenous success probabilities. Therefore, some but by no means all numerical specifications of our model will fit into the Goyal and Joshi classification.

# 5 Concluding Remarks

The model developed here as well as a substantial part of the network literature is concerned with information flows. Such models may be interpreted as a reduced form where all costs and benefits have been attributed to information flows. Under perfect reliability, the primary focus lies on the size and efficiency of networks. With imperfect reliability the strength of social ties, or the nature and quality of information can be discussed as well. In our model for instance, one could argue that the information exchange between $i$ and $j$ is valuable with probability $p_{ij}$ and is of a dubious nature with the complementary probability. Thus, imperfect reliability raises questions about a possible quantity-quality trade-off as well as the related efficiency issues.

The assumption of agent heterogeneity in the form of imperfect reliability in social networks provides a richer set of results than the homogeneous setting. In conjunction with our adopted solution concept, Nash equilibrium, it accentuates open questions that also arise − though perhaps to a lesser degree − in the context of pairwise stability. An example is the issue of cost sharing and side payments. Twice in the course of our current investigation we came across this issue: First, in the discussion of efficiency and Pareto-optimality. For a second time in the context of the mutual consent model. The issue of cost sharing and bargaining over the costs of link formation is especially crucial when benefits mainly accrue from indirect links. It indicates an important direction for future research. Currarini and Morelli (2000) take a first step in this direction. They introduce a noncooperative game of sequential network formation in which players propose links and demand payoffs. They show that for networks which satisfy size monotonicity, there is no conflict between efficiency and stability.

Bala and Goyal's work on Nash networks shows that results under imperfect reliability are quite different from those in a deterministic setting. With the introduction of heterogeneity this clear distinction no longer prevails. Our findings encompass results of both types of models. For example, with perfect reliability and information decay, non-empty Nash networks are always minimally connected, irrespective of the size of society (Bala and Goyal (2000a)). In contrast, with homogeneous imperfect reliability and no information decay, redundant links between agents always arise asymptotically (Bala and Goyal (2000b)). In our model, with heterogeneous imperfect reliability and no information decay, both types of outcomes can be generated through appropriate choice of the $p_{ij}$'s. For instance, decay models (with perfect reliability) compute benefits by considering only the shortest path between agents. Extra indirect links do not contribute to benefits. Given a resulting minimally connected Nash network $g$ of such a model, there exists a parameter specification of our model that also gives rise to $g$ as a Nash network. In our framework this requires lowering the $p_{ij}$ to zero or below $p_0$ for all $ij$ with $g_{ij} = 0$ and $g_{ji} = 0$ and choosing sufficiently high probabilities $p_{ij}$ for all other $ij$ so that all benefits accrue from the direct links only. On the other hand, as discussed in subsection 3.1, choosing the $p_{ij}$'s appropriately leads to super-connected networks as well.

The four alternative model specifications introduced here provide immediate results as well as valuable insights for future work. The double links model which is the most thoroughly investigated formulation allows three important conclusions. First note that double links are not likely to be established when probabilities are very high or very low. Next notice that because of the increase in link reliability, certain high link establishment costs can now be surmounted allowing creation of new links. Finally observe that the nature of super-connected networks changes since agents might prefer to reinforce high probability links over forming new links. The second model variation demonstrates that under incomplete information, some of the previous Nash networks become quite fragile. Our third model variation clearly indicates that costless mutual consent will only lead to a larger set of equilibria and trivially guarantees existence of equilibria. It is evident that consent models must incorporate costly consent to yield further insights. Our last model specification shows that negative link externalities — which manifest themselves in endogenous success probabilities — can yield very fragmented Nash networks. They can cause the disappearance of stars and the emergence of other interesting network architectures.

Finally, to end on a cautionary note, we have indicated the possibility of network failure in the discussion following Proposition 6. It is only appropriate to mention Greif's (1994) tale of two historical societies – the Maghribi traders, with an Islamic culture who shared trading information widely, and the Genoese traders exemplifying the Latin world, who operated individually and did not share information amongst each other, relying more on legal contracts. He argues that the culture and social organization of these two communities ultimately determined their long-run survival. The Genoese kept business secrets from each other, improved their contract law and operated through the market. Consequently they ended up with an efficient society. The Maghribis on the other hand operated through an informal network where behavior of a single pair of agents affected everyone in the network. As opposed to the Genoese traders the Maghribis invested considerable time and effort to gather information about their network. Since one bad link could adversely affect the entire network, the Maghribis often had to engage in superfluous links as well without adequate concern for efficiency. Efficiency became a critical issue once new business opportunities arose in far away lands, where operating through an ethnically based network became very expensive. In the end these organizational differences created by the cultural beliefs of the two societies led to the survival of the more efficient of the two. Thus social networks may be good substitutes for anonymous markets in certain societies, but the market paired with the proper infrastructure may be a more efficient institution. In fact for trade in standardized commodities, a frictionless and informationally efficient anonymous market, if feasible, would be best. Some of the trade-offs between networks and anonymous markets are addressed by Kranton (1996) who investigates the persistence and coexistence of personalized exchange arrangements when anonymous market channels are available and would be more efficient.

# 6 Appendix: Proofs

1. *Proof of Proposition 5*:

Consider the outward pointing star. All agents $m \in M$ have a link with the central agent, and the conditions for not deviating from the Nash strategy identified in *Proposition 3* remain unchanged. However, we must also verify that neither agent $n$ nor any $m \in M$ will gain by adding a link. For all $m \in M$ and $k \in K_m$ we need to compute $\Pi_m(g^{ot} + g_{mk})$ which is the sum of payoffs from three different terms: the payoff related to player $n$, the payoff related to player $k$, and the payoff from links to all others players $j \in J_k$.

- The payoff related to player $n$ is given by
  $r'_{nm} \equiv p_{nm}(1 - p_{mk}p_{nk}) + (1 - p_{nm})p_{mk}p_{nk} + p_{nm}p_{mk}p_{nk}$.
- The payoff related to player $k$ is given by
  $r'_{mk} \equiv p_{mk}(1 - p_{nm}p_{nk}) + (1 - p_{mk})p_{nm}p_{nk} + p_{nm}p_{mk}p_{nk}$.
- Finally, the payoff from all other players is given by $r'_{nm}\Sigma_k$.

Adding all these up yields

$$\Pi_m(g^{ot} \oplus g_{mk}) = r'_{nm} + r'_{mk} + r'_{nm}\Sigma_k.$$

The link $mk$ will not be formed when $\Pi_m(g^{ot} \oplus g_{mk}) - \Pi_m(g^{ot}) < 0$, or $(1 - p_{nm})p_{mk}p_{nk} + p_{mk}(1 - p_{nm}p_{nk}) + (1 - p_{nm})p_{mk}p_{nk}\Sigma_k < c$. Choose the threshold probability $\delta_k^m$ as the smallest number such that this inequality holds, if $p_{ij} > \delta_k^m$ for all $i \neq j$. Regarding agent $n$, he does not want to form an additional link with $m \in M$, if

$$p_{mn}(1 - p_{mn}) < c.$$

But this condition follows from $p_{nm} > \delta_n^m$, where $\delta_n^m$ is the smallest number such that the inequality $p_{nm}(1 - p_{nm}) + p_{nm}(1 - p_{nm})\sum_{k \in K_m} p_{nk} < c$ holds. Finally, set $\widetilde{\delta} = \max_{m \in M} \max_{i \neq m} \delta_i^m$. Then if $p_{ij} \in (\widetilde{\delta}, 1)$ for all $ij$, we can support the outward pointing star as Nash. ■

2. *Proof of Proposition 6*:

Consider an outward pointing star. The central agent $n$ plays no role in this case. Every agent $m$ receives a perceived payoff of $\Pi_m(g^{ot}) = p_{mn} + (n-2)p_{mn}p_m - c$. Consider the possibility that agent $m$ wants to deviate and form a link with some $k \in K_m$. Her payoffs from this are given by $\Pi_m(g^{ot} + g_{mk} - g_{mn}) = p_{mk} + p_{mk}p_m + (n-3)p_{mk}(p_m)^2 - c$. Hence the condition for no deviation is given by

$$p_{mn} - p_{mk} + p_m(p_{mn} - p_{mk}) + (n-3)(p_{mn} - p_m p_{mk})p_m > 0$$

which is true when the modified star condition holds. In order to rule out additional links, we require that $(n-2)(1 - p_{nm}p_m) < ((n-2) - p_{nm}\sum_{k \in K_m} p_{nk})$. Then the analogue of **(2)** holds if $p_{ij} \in (\delta, 1)$ for all $ij$. This completes the proof. ■

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